

Random Transverse Field Ising Model in dimension $d = 2, 3$: Infinite Disorder scaling via a non-linear transfer approach

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The 'Cavity-Mean-Field' approximation developed for the Random Transverse Field Ising Model on the Cayley tree [L. Ioffe and M. Mézard, PRL 105, 037001 (2010)] has been found to reproduce the known exact result for the surface magnetization in $d = 1$ [O. Dimitrova and M. Mézard, J. Stat. Mech. (2011) P01020]. In the present paper, we propose to extend these ideas in finite dimensions $d > 1$ via a non-linear transfer approach for the surface magnetization. In the disordered phase, the linearization of the transfer equations correspond to the transfer matrix for a Directed Polymer in a random medium of transverse dimension $D = d - 1$, in agreement with the leading order perturbative scaling analysis [C. Monthus and T. Garel, arXiv:1110.3145]. We present numerical results of the non-linear transfer approach in dimensions $d = 2$ and $d = 3$. In both cases, we find that the critical point is governed by Infinite Disorder scaling. In particular exactly at criticality, the one-point surface magnetization scales as $\ln m_L^{surf} \simeq -L^{\omega_c} v$, where $\omega_c(d)$ coincides with the droplet exponent $\omega_{DP}(D = d - 1)$ of the corresponding Directed Polymer model, with $\omega_c(d = 2) = 1/3$ and $\omega_c(d = 3) \simeq 0.24$. The distribution $P(v)$ of the positive random variable v of order $O(1)$ presents a power-law singularity near the origin $P(v) \propto v^a$ with $a(d = 2, 3) > 0$ so that all moments of the surface magnetization are governed by the same power-law decay $(m_L^{surf})^k \propto L^{-x_s}$ with $x_s = \omega_c(1 + a)$ independently of the order k .

I. INTRODUCTION

The quantum Ising model

$$\mathcal{H} = - \sum_{\langle i,j \rangle} J_{i,j} \sigma_i^z \sigma_j^z - \sum_i h_i \sigma_i^x \quad (1)$$

where the nearest-neighbor couplings $J_{i,j} > 0$ and the transverse-fields $h_i > 0$ are independent random variables drawn with two distributions $\pi_{coupling}(J)$ and $\pi_{field}(h)$ is the basic model to study quantum phase transitions at zero-temperature in the presence of frozen disorder. In dimension $d = 1$, exact results for a large number of observables have been obtained by Daniel Fisher [1] via the asymptotically exact Strong Disorder renormalization procedure (for a review, see [2]). In particular, the transition is governed by an Infinite Disorder fixed point and presents unconventional scaling laws with respect to the pure case. In dimension $d > 1$, the Strong Disorder renormalization procedure can still be defined. It cannot be solved analytically, because the topology of the lattice changes upon renormalization, but it has been studied numerically with the conclusion that the transition is also governed by an Infinite Disorder fixed point in dimensions $d = 2, 3, 4$ [3–13]. These numerical renormalization results are in agreement with the results of independent quantum Monte-Carlo in $d = 2$ [14, 15].

Even if it is clear that the most natural method to study Infinite Disorder fixed points is the Strong Disorder renormalization approach, it seems useful to determine whether other approaches are able to describe Infinite Disorder scaling. In this paper, we introduce a simple non-linear transfer approximation for the surface magnetization in finite dimension $d > 1$, which is inspired from the 'Cavity-Mean-Field' approximation developed in Refs [16–18], and we study numerically the critical properties of this approximation in dimensions $d = 2$ and $d = 3$.

The paper is organized as follows. In Section II, we recall briefly the 'Cavity-Mean-Field' approximation developed in Refs [16–18] and introduce the non-linear transfer approach for finite dimensions $d > 1$. Our numerical results in dimension $d = 2$ and $d = 3$ are presented in sections III and IV respectively. Our conclusions are summarized in section V.

II. NON-LINEAR TRANSFER APPROACH FOR THE SURFACE MAGNETIZATION

A. 'Cavity-Mean-Field' approximation on the Cayley tree [16–18]

For the random quantum Ising model model defined on a tree of coordinence $(K + 1)$, the following 'Cavity-Mean-Field' approximation has been developed [16–18] : an ancestor i is submitted to the effective single spin Hamiltonian

$$H_i^{eff} = -B_i \sigma_i^z - h_i \sigma_i^x \quad (2)$$

where h_i is its own random transverse field, and where B_i represents the longitudinal field created by the K children j (related to i by the ferromagnetic couplings J_{ij}) within a 'Mean-Field approximation' (the operator σ_j^z is replaced by its expectation value $\langle \sigma_j^z \rangle$)

$$B_i = \sum_{j=1}^K J_{i,j} \langle \sigma_j^z \rangle \quad (3)$$

The effective Hamiltonian of Eq. 2 is only a two-level system that can be solved immediately : the magnetization of the ground state reads

$$m_i \equiv \langle \sigma_i^z \rangle_{H_i^{eff}} = \frac{B_i}{\sqrt{B_i^2 + h_i^2}} \quad (4)$$

Using Eq. 3, one obtains the following non-linear recurrence for the magnetizations m_i (see Eq. 4 of [16], Eq. 7 of [17], Eq. 17 of [18] in the limit of zero temperature $\beta = +\infty$)

$$m_i = \frac{\sum_{j=1}^K J_{i,j} m_j}{\sqrt{\left(\sum_{j=1}^K J_{i,j} m_j\right)^2 + h_i^2}} \quad (5)$$

We refer to Refs [16–18] for more details on this 'Cavity-Mean-Field' approximation and on its properties. As a final remark, let us stress that the 'Cavity-Mean-Field' is not exact for the pure model on the Cayley tree (see Fig. 3 of Ref [18]), but has been argued to become quantitatively correct in the limit of high connectivity $K \gg 1$ [16–18].

In the disordered phase where the magnetizations flows towards zero, the non-linear recurrence of Eq. 5 can be linearized to give the following recursion

$$m_i \simeq \frac{1}{h_i} \sum_{j=1}^K J_{i,j} m_j \quad (6)$$

which is equivalent to the problem of a Directed Polymer on the Cayley tree [16–18]. This equivalence can be justified directly at the level of lowest-order perturbation theory (i.e. without invoking the 'Cavity-Mean-Field' approximation of Eq. 5), and can be in this way extended to the finite-dimensional case [19].

B. 'Cavity-Mean-Field' approximation in $d = 1$ [18]

For $K = 1$, the Cayley tree of coordination $(K + 1)$ discussed in the previous section becomes a one-dimensional chain, and Eq. 5 becomes the one-dimensional non-linear recurrence [18]

$$m_i = \frac{J_{i,i+1} m_{i+1}}{\sqrt{(J_{i,i+1} m_{i+1})^2 + h_i^2}} \quad (7)$$

Assuming one starts with the boundary condition $m_L = 1$ at site $i = L$, one obtains the following explicit expression for the surface magnetization m_0^{surf} at the site $i = 0$ [18]

$$m_0^{surf} = \left[1 + \sum_{i=0}^{L-1} \prod_{j=0}^i \left(\frac{h_j}{J_{j,j+1}} \right)^2 \right]^{-1/2} \quad (8)$$

As stressed in [18], this expression exactly coincides with the rigorous expression that can be obtained from a free-fermion representation [20, 21], and from which many critical exponents can be obtained [21–23].

The reason why the 'Cavity-Mean-Field' approximation turns out to become exact for the surface magnetization in $d = 1$ is not clear to us, and seems rather surprising : usually 'mean-field approximation' are exact in sufficiently high dimensions or on trees, and are not exact in low dimensions, the 'worst case' being precisely $d = 1$. Here we have exactly the opposite conclusion : the 'Cavity-Mean-Field' is not exact for the pure model on the tree (for the disordered case, it is not known), but turns out to be exact in $d = 1$, both for the pure and the disordered case. In the absence of any satisfactory explanation for this unusual situation, we tend to think that the exactness of the 'Cavity-Mean-Field' in $d = 1$ is likely to be a 'coincidence' specific to this particular case, from which one cannot draw general conclusions for the validity of this approach in higher dimensions or for other quantum disordered models.

2. Linearized transfer matrix within the disordered phase

Within the disordered phase, the surface magnetizations $m(x = L, y)$ are expected to decay typically exponentially in L , so that one may linearize the transfer Eqs 10 and 11 to obtain

$$\textbf{Linearization : } m(x, y) \simeq \frac{J_{\{(x,y), (x-1, y+\frac{1}{2})\}}}{h(x, y)} m\left(x-1, y+\frac{1}{2}\right) + \frac{J_{\{(x,y), (x-1, y-\frac{1}{2})\}}}{h(x, y)} m\left(x-1, y-\frac{1}{2}\right) \quad (12)$$

This linearized equations can be derived directly within a lowest-order perturbative approach [19] (i.e. without invoking the 'Cavity-Mean-Field' approximation) and corresponds to the transfer matrix satisfied by the partition function of a Directed Polymer with $D = (d-1)$ transverse directions, as discussed in detail in [19]. We refer to [19] for the description of the consequences of this correspondence, and for the analogy with Anderson localization, where the droplet exponent of the Directed Polymer also appears in the localized phase [24–26]. Here our conclusion is that the non-linear transfer approach describes at least correctly the disordered phase, where it coincides with the lowest-order perturbative approach [19].

3. Discussion

Besides its correctness in the disordered phase that we have just discussed, the validity of the non-linear transfer exactly at criticality and in the ordered phase has to be studied for the disordered case in $d > 1$. Since it has been found to be exact in $d = 1$ (see section IIB), one could hope that it is not 'too bad' in $d = 2, 3$ (even if it is clear that this approximation is not valid for the pure model) : we believe that it should capture correctly the nature of the transition between 'Infinite-Disorder' or 'Conventional' scaling. In the following, we present our numerical results in $d = 2$ and $d = 3$ and discuss the scaling properties in the two phases and at criticality.

III. NUMERICAL RESULTS IN DIMENSION $d = 2$

In this section, we present the numerical results obtained with the following sizes L and the corresponding numbers $n_s(L)$ of disordered samples of volume L^2

$$\begin{aligned} L &= 10^3, 2.10^3, 3.10^3, 4.10^3, 5.10^3, 6.10^3, 7.10^3, 8.10^3 \\ n_s(L) &= 2.10^5, 13.10^4, 65.10^3, 37.10^3, 24.10^3, 17.10^3, 13.10^3, 10^4 \end{aligned} \quad (13)$$

For each sample α , we collect the L values of the surface magnetization $m^{(\alpha)}(x = L, i)$ at the different points $i = 1, 2, \dots, L$ of the surface (see Fig. 1). Average values and histograms are then based on these $L \times n_s(L)$ values.

We have chosen to consider the following log-normal distribution for the random transverse fields $h_i > 0$

$$\pi_{LN}(h) = \frac{1}{h\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln h - \overline{\ln h})^2}{2\sigma^2}} \quad (14)$$

of parameter $\overline{\ln h} = 0$ and $\sigma = 1$, whereas the ferromagnetic couplings $J_{i,j}$ are not random but take a single value J that will be the control parameter of the quantum transition.

A. Disordered phase ($J < J_c$)

1. Exponential decay of the typical surface magnetization $m_L^{typ} \equiv e^{\overline{\ln m_L^{surf}}}$

In the disordered phase $J < J_c$, one expects that the typical surface magnetization defined by

$$\ln(m_L^{typ}) \equiv \overline{\ln m_L^{surf}} \quad (15)$$

decays exponentially with L

$$\ln(m_L^{typ}) \equiv \overline{\ln m_L^{surf}}(J < J_c) \underset{L \rightarrow \infty}{\simeq} -\frac{L}{\xi_{typ}(J)} \quad (16)$$

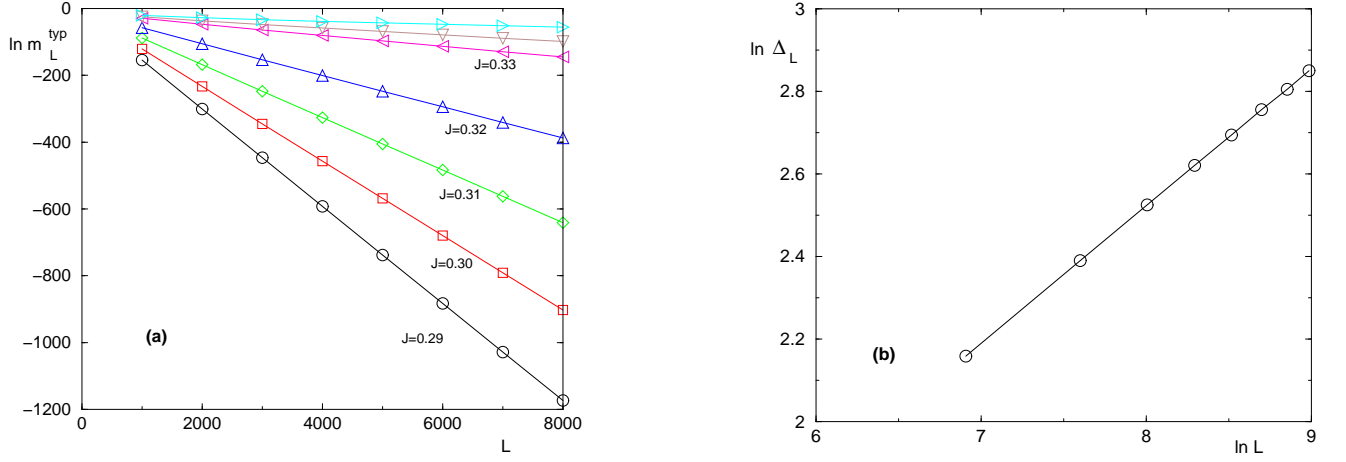


FIG. 2: Disordered phase $J < J_c$ in $d = 2$ (a) Exponential decay of the typical surface magnetization $m_L^{typ} \equiv e^{\overline{\ln m_L^{surf}}}$: we show the linear decay of $\ln m_L^{typ}$ as a function of the length L (see Eq. 16). (b) Log-log plot of the width Δ_L of the distribution of the logarithm of the surface magnetization as a function of L (here for $J = 0.29$): the slope is of order $\omega \simeq 0.33$ (see Eq. 20).

where ξ_{typ} represents the typical correlation length that diverges at the transition as a power-law

$$\xi_{typ}(J) \underset{J \rightarrow J_c^-}{\simeq} (J_c - J)^{-\nu_{typ}} \quad (17)$$

On Fig. 2 (a) we show our numerical results: concerning the exponential decay with L of Eq. 16 for various values of J . We find that the corresponding slope $1/\xi_{typ}(J)$ vanishes near the critical value $J_c \simeq 0.335$ with the exponent

$$\nu_{typ} \simeq 1 \quad (18)$$

2. Growth of the width of the distribution of the logarithm of the surface magnetization

In the disordered phase $J < J_c$, one expects that the width Δ_L of the distribution of the logarithm of the surface magnetization defined by

$$\Delta_L \equiv \left(\overline{(\ln m_L^{surf})^2} - (\overline{\ln m_L^{surf}})^2 \right)^{1/2} \quad (19)$$

grows as a power-law of L

$$\Delta_L(J < J_c) \underset{L \rightarrow \infty}{\simeq} L^\omega \quad (20)$$

Our numerical data shown on Fig. 2 (b) correspond to the value

$$\omega(d = 2) \simeq 0.33 \quad (21)$$

in agreement with the argument presented in [19] where $\omega(d = 2)$ should coincide with the Directed Polymer droplet exponent $\omega_{DP}(D = d - 1 = 1) = 1/3$ [27–30].

3. Distribution of the logarithm of surface magnetization

We show on Fig 3 (a) our numerical results concerning histograms of the logarithm of the surface magnetization in the disordered phase. Our conclusion is that the surface magnetization follows the scaling

$$\ln(m_L^{surf}) \underset{L \rightarrow \infty}{\simeq} \ln(m_L^{typ}) + \Delta_L u \quad (22)$$

where the behaviors of the typical value $\ln(m_L^{typ}) \simeq -L/\xi_{typ}$ and of the width $\Delta_L \sim L^\omega$ have been already discussed above in Eqs 16 and 20 respectively. On Fig. 3 (b), we show that the stable distribution $P(u)$ of the rescaled variable u coincides with the GOE Tracy-Widom distribution, as expected from the correspondence with the Directed Polymer model in the disordered phase [19].

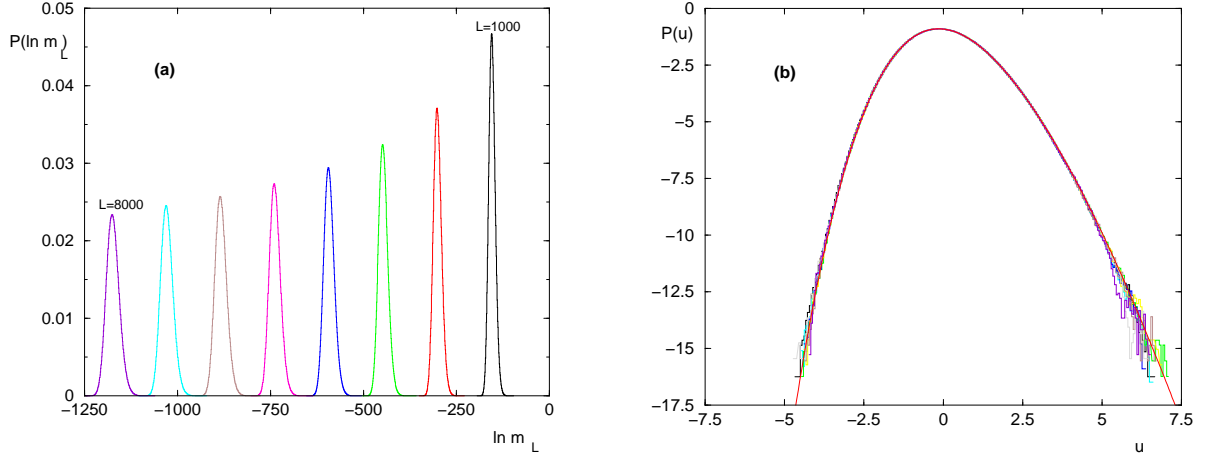


FIG. 3: Disordered phase in $d = 2$ (here $J = 0.29$) : (a) Evolution with L of the probability distribution $P_L(\ln m_L^{surf})$ of the logarithm of surface magnetization : (b) Corresponding fixed distribution of the rescaled variable $u = (\ln m_L^{surf} - \ln m_L^{typ}) / \Delta_L$ in log-scale to show the tails, compared to the exact Tracy-Widom GOE distribution (thick line).

B. Ordered phase

1. Behavior of the typical surface magnetization m_∞^{typ} in the ordered phase $J > J_c$

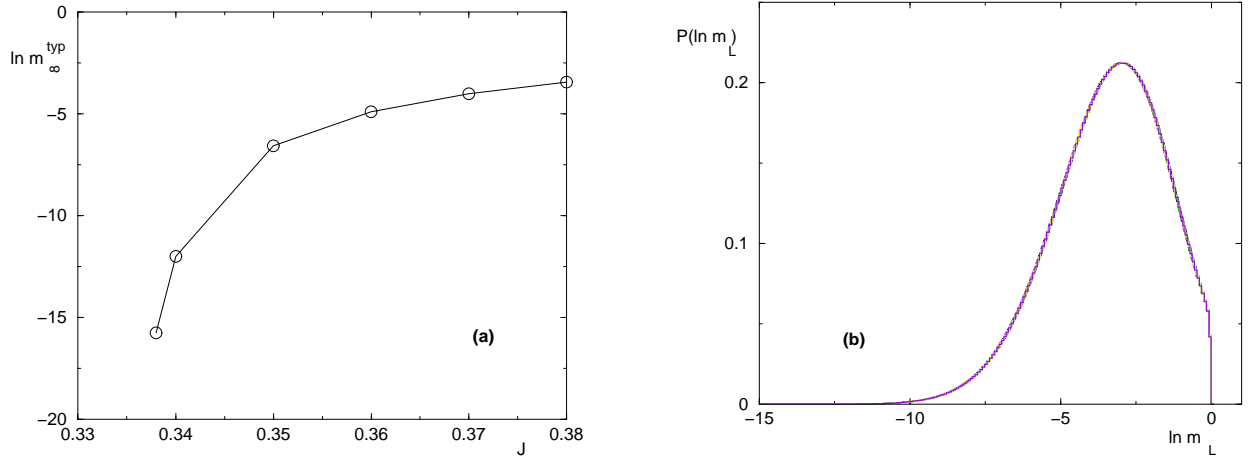


FIG. 4: Ordered Phase $J > J_c$ in $d = 2$: (a) Behavior of the asymptotic typical surface magnetization m_∞^{typ} as a function of the ferromagnetic coupling J : our numerical data are compatible with an essential singularity (Eq. 24) of exponent $\kappa \simeq 0.5$ (b) the probability distribution $P_L(\ln m_L^{surf})$ (here for $J = 0.38$) of the logarithm of surface magnetization remains fixed and attached at the origin (as a consequence of the bound $m_L^{surf} \leq 1$).

In the ordered phase, the typical surface magnetization remains finite in the limit where the number of generations L diverges

$$\ln m_L^{typ}(J > J_c) \equiv \overline{\ln m_L^{surf}(J > J_c)}_{L \rightarrow \infty} \simeq \ln m_\infty(J > J_c) > -\infty \quad (23)$$

and one expects an essential singularity behavior

$$\ln m_\infty^{typ}(J > J_c) \underset{J \rightarrow J_c^+}{\propto} -(J - J_c)^{-\kappa} \quad (24)$$

Our data shown on Fig. 4 (a) can be fitted with the value

$$\kappa(d = 2) \simeq 0.5 \quad (25)$$

that can be related to other exponents via finite-size scaling (see below around Eq. 34)

2. Distribution of the logarithm of the surface magnetization

In the ordered phase, the probability distribution $P_L(\ln m_L^{surf})$ of the logarithm of surface magnetization remains fixed as L varies, and terminates discontinuously at the origin, as a consequence of the bound $m_L^{surf} \leq 1$ corresponding to $\ln m_L^{surf} \leq 0$ (see Fig. 4 (b))

C. Critical point

1. Behavior of the typical surface magnetization m_L^{typ} and of the width Δ_L

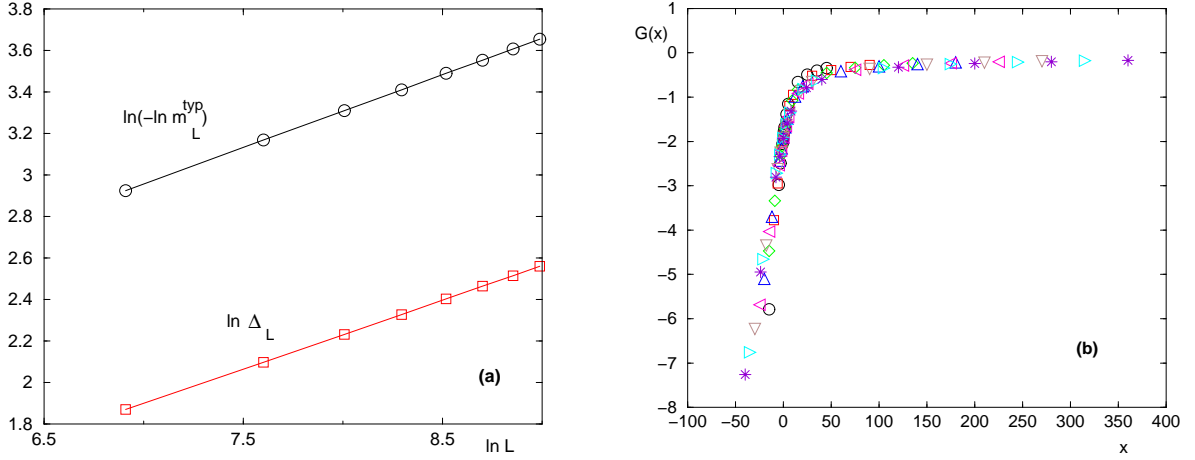


FIG. 5: Critical point (here $J_c = 0.335$) : (a) Log-log plot of the logarithm of the typical surface magnetization m_L^{typ} and of the width Δ_L : both slopes are of order $\omega_c \simeq 0.33$ (see Eqs 26 and 27) (b) Finite-size scaling of the typical surface magnetization according to Eq. 29 with $J_c = 0.335$, $\omega_c = 0.33$ and $\nu_{av} = 1.5$.

Exactly at criticality, one expects that the typical surface magnetization follows an activated behavior of exponent $\omega_c < 1$ (compare with Eq. 16 in the disordered phase)

$$\ln(m_L^{typ}(J = J_c)) \equiv \overline{\ln m_L^{surf}(J = J_c)} \underset{L \rightarrow \infty}{\simeq} -L^{\omega_c} \quad (26)$$

and that the width defined in Eq. 19 is also governed by the same exponent

$$\Delta_L(J = J_c) \underset{L \rightarrow \infty}{\simeq} L^{\omega_c} \quad (27)$$

Our numerical data at $J_c \simeq 0.335$ shown on Fig. 5 (a) are compatible with these behaviors with the value

$$\omega_c \simeq 0.33 \quad (28)$$

i.e. ω_c coincides with the fluctuation exponent ω measured in the disordered phase (see Eq. 21)

This last property implies that the finite-size scaling for the typical surface magnetization m_L^{typ} involves some correlation length exponent ν_{av} different from ν_{typ}

$$\ln m_L^{typ}(J) \equiv \overline{\ln m_L^{surf}(J)} \simeq -L^{\omega_c} G\left(x \equiv (J - J_c)L^{1/\nu_{av}}\right) \quad (29)$$

The matching with the behavior of Eq. 22 in the disordered phase implies that

$$G(x) \underset{x \rightarrow -\infty}{\propto} (-x)^{\nu_{typ}} \quad (30)$$

and that ν_{av} reads

$$\nu_{av} = \frac{\nu_{typ}}{1 - \omega} \quad (31)$$

This relation can be understood within a rare events analysis for the averaged correlation in the disordered phase [19]. The values $\nu_{typ} \simeq 1$ and $\omega = 1/3$ yield

$$\nu_{av}(d=2) \simeq \frac{3}{2} \quad (32)$$

The matching of the finite-size scaling form of Eq. 29 with the essential singularity of Eq. 24 in the ordered phase implies that

$$G(x) \underset{x \rightarrow +\infty}{\propto} \frac{1}{x^\kappa} \quad (33)$$

with

$$\kappa = \omega_c \nu_{av} = \nu_{typ} \frac{\omega}{1 - \omega} \quad (34)$$

The values $\nu_{typ} = 1$ and $\omega = 1/3$ yield

$$\kappa(d=2) \simeq \frac{1}{2} \quad (35)$$

in agreement with the estimate of Eq. 25.

As shown on Fig 5 (b), our numerical data collapse well with the finite-size scaling form of Eq. 29 with $\nu_{av} = 1.5$.

2. Distribution of the logarithm of surface magnetization

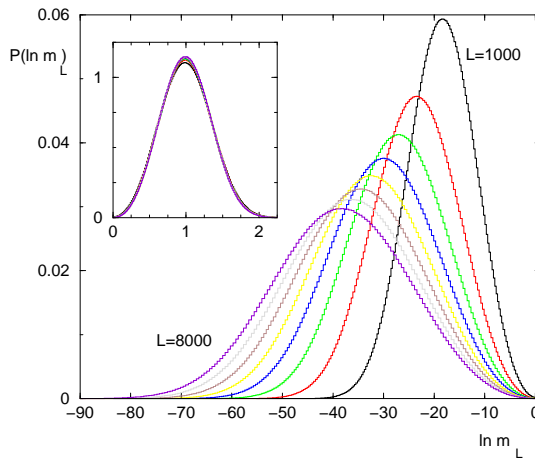


FIG. 6: Critical point in $d = 2$ (here $J_c = 0.335$) : Evolution with L of the probability distribution $P_L(\ln m_L^{surf})$ of the logarithm of surface magnetization. Inset : Corresponding fixed distribution of the rescaled variable $v = (\ln m_L^{surf}) / \ln m_L^{typ}$.

At criticality, the rescaled variable

$$v \equiv \frac{\ln m_L^{surf}}{\ln m_L^{typ}} \propto -\frac{\ln m_L^{surf}}{L^{\omega_c}} \quad (36)$$

remains a positive random variable of order $O(1)$ as $L \rightarrow +\infty$. Our numerical measure of its probability distribution $P(v)$ shown on Fig. 6 is compatible with a power-law singularity near the origin

$$P(v) \underset{v \rightarrow 0^+}{\simeq} v^a \quad (37)$$

with an exponent of order $a \geq 2$ that we do not measure precisely. Note that this is different from the case $d = 1$ where $P(v = 0)$ is finite ($a = 0$). We have not been able to find a physical argument to predict the value of a in $d = 2$. This exponent a will directly influence the scaling of the moments of the surface magnetization, as we now discuss.

3. Moments of the surface magnetization

In contrast to the activated behavior of the typical surface magnetization m_L^{typ} of Eq. 26, the moments of the surface magnetization are expected to follow a power-law, as a consequence of the following rare events analysis : the surface magnetization of Eq. 36 will be of order $O(1)$ if the random variable v happens to be smaller than $1/L^{\omega_c}$. Taking into account the behavior of Eq. 37, this will happen with probability

$$Prob(m_L^{surf} = 1) \simeq \int_0^{1/L^{\omega_c}} dv P(v) \sim \int_0^{1/L^{\omega_c}} dv v^a \underset{L \rightarrow \infty}{\propto} L^{-\omega_c(1+a)} \quad (38)$$

and all moments will be governed by this power-law

$$\overline{(m_L^{surf})^k} \simeq Prob(m_L^{surf} = 1) \underset{L \rightarrow \infty}{\propto} L^{-x_s} \quad \text{with} \quad x_s = \omega_c(1+a) \quad (39)$$

independently of the order k .

Our numerical data for the three first moments $k = 1, 2, 3$ and various sizes are compatible with Eq. 39 with an exponent

$$x_s(d=2) \simeq 1.2 \quad (40)$$

The relation of Eq. 39 then corresponds to

$$a(d=2) \simeq 2.6 \quad (41)$$

In the ordered phase, our numerical data are compatible with the power-law

$$\overline{m_L^{surf}} \propto (J - J_c)^{\beta_s} \quad (42)$$

with

$$\beta_s(d=2) = x_s \nu_{av} \simeq 1.8 \quad (43)$$

IV. NUMERICAL RESULTS IN DIMENSION $d = 3$

In this section, we present the numerical results obtained with the following sizes L and the corresponding numbers $n_s(L)$ of disordered samples of volume L^3

$$\begin{aligned} L &= 10^2, 2 \cdot 10^2, 3 \cdot 10^2, 4 \cdot 10^2, 5 \cdot 10^2, 6 \cdot 10^2, 7 \cdot 10^2, 8 \cdot 10^2 \\ n_s(L) &= 27 \cdot 10^3, 7 \cdot 10^3, 3 \cdot 10^3, 16 \cdot 10^2, 10^3, 7 \cdot 10^2, 5 \cdot 10^2, 4 \cdot 10^2 \end{aligned} \quad (44)$$

For each sample α , we collect the L^2 values of the surface magnetization at the different points of the surface. Average values and histograms are then based on these $L^2 \times n_s(L)$ values. We consider again the disorder distribution of Eq. 14 and take J as the control parameter of the transition.

A. Disordered phase

Our data follow the scaling of Eq. 22, with the following properties :

(i) the scaling of the typical surface magnetization is given by Eq 16, and the typical correlation length exponent of Eq. 17, seems again very close to unity

$$\nu_{typ} \simeq 1 \quad (45)$$

(ii) the width Δ_L of Eq. 19 grows as the power-law of Eq. 20 with the exponent

$$\omega(d=3) \simeq 0.24 \quad (46)$$

that coincides with the numerical values of the droplet exponent of the Directed Polymer model with $D = d - 1 = 2$ transverse dimensions [31–37], in agreement with the argument presented in [19].

(iii) As L grows, the evolution of the probability distribution $P_L(\ln m_L^{surf})$ is shown on Fig. 7 (a). The corresponding fixed distribution of the rescaled random variable $u = (\ln m_L^{surf} - \ln m_L^{typ})/\Delta_L$ is shown on Fig 7 (b).

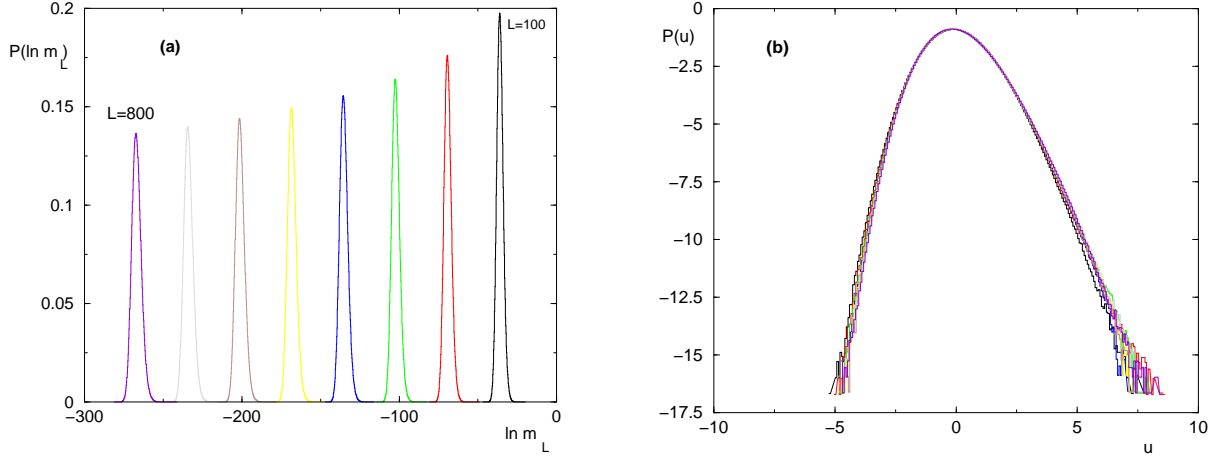


FIG. 7: Disordered phase in $d = 3$ (here $J = 0.11$) : (a) Evolution with L of the probability distribution $P_L(\ln m_L^{surf})$ of the logarithm of surface magnetization : (b) Corresponding fixed distribution of the rescaled variable $u = (\ln m_L^{surf} - \ln m_L^{typ}) / \Delta_L$ in log-scale to show the tails.

B. Critical point

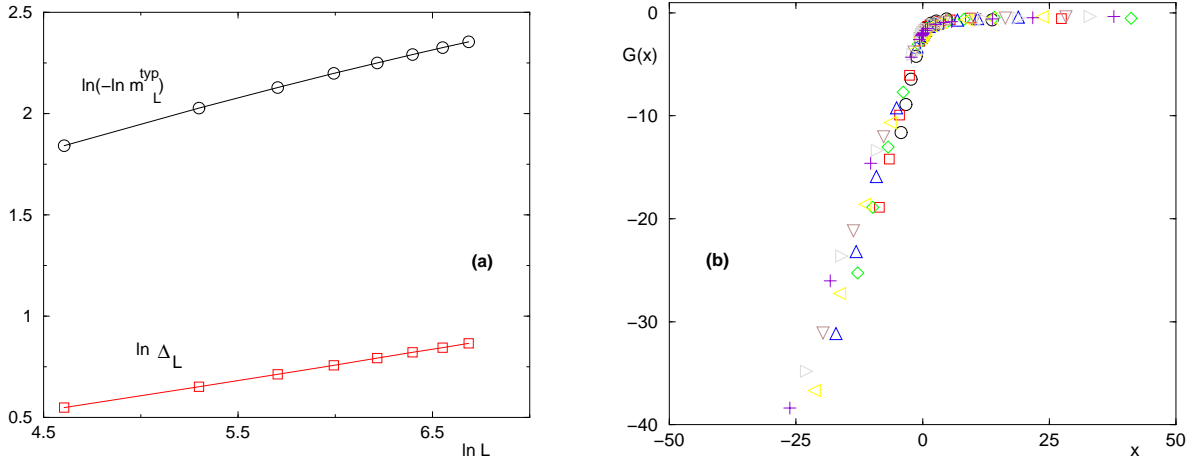


FIG. 8: Typical surface magnetization in the critical region in $d = 3$ (a) Log-log plot of the logarithm of the typical surface magnetization m_L^{typ} and of the width Δ_L at criticality $J_c = 0.1528$: both slopes are of order $\omega_c \simeq 0.24$ (see Eqs 26 and 27) (b) Finite-size scaling of the typical surface magnetization in $d = 3$ according to Eq. 29 with $\nu_{av} = 1.32$.

At criticality $J_c = 0.1528$, we find that the exponent ω_c of Eqs 26 and 27 coincides with the value of ω of Eq. 46 concerning the disordered phase (see Fig. 8 (a))

$$\omega_c(d = 3) \simeq 0.24 \quad (47)$$

We show on Fig. 8 (b) the finite-size scaling analysis of Eq. 29 for the logarithm of the typical surface magnetization with the averaged correlation length exponent $\nu_{av} = 1/(1 - \omega) \simeq 1.32$.

We show on Fig. 9 our numerical data for the probability distribution of the surface magnetization at criticality : the fixed point distribution $P(v)$ of the rescaled variable v of Eq. 36 displays a power-law singularity near the origin (Eq 37), that will determine the scaling of all moments of the surface magnetization according to Eq. 39. Our numerical data for the moments are compatible with Eq. 39 with an exponent

$$x_s(d = 3) \simeq 1.34 \quad (48)$$

that would correspond to

$$a(d = 3) = \frac{x_s}{\omega_c} - 1 \simeq 4.5 \quad (49)$$

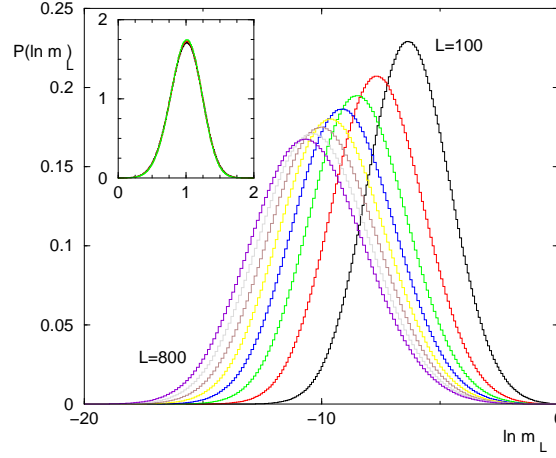


FIG. 9: Critical point in $d = 3$ (here $J_c = 0.1528$) : Evolution with L of the probability distribution $P_L(\ln m_L^{surf})$ of the logarithm of surface magnetization. Inset : Corresponding fixed distribution of the rescaled variable $v = (\ln m_L^{surf}) / \ln m_L^{typ}$.

and to the exponent (Eq. 42)

$$\beta_s(d=3) = x_s \nu_{av} \simeq 1.76 \quad (50)$$

V. CONCLUSION

Since the 'Cavity-Mean-Field' approximation developed for the Random Transverse Field Ising Model on the Cayley tree [16–18] has been found to reproduce the known exact result for the surface magnetization in $d = 1$ [18], we have proposed to extend these ideas in finite dimensions $d > 1$ via a non-linear transfer approach for the surface magnetization. In the disordered phase, the linearization (Eq 12) of the transfer equations correspond to the transfer matrix for a Directed Polymer in a random medium of transverse dimension $D = d - 1$, in agreement with the leading order perturbative scaling analysis [19].

We have presented numerical results of this non-linear transfer approach in dimensions $d = 2$ and $d = 3$, where large system sizes can be easily studied. In both cases, we have found that the critical point is governed by Infinite Disorder scaling. In particular exactly at criticality, the one-point surface magnetization scales as $\ln m_L^{surf} \simeq -L^{\omega_c} v$, where $\omega_c(d)$ coincides with the droplet exponent $\omega_{DP}(D = d - 1)$ of the corresponding Directed Polymer model, with $\omega_c(d = 2) = 1/3$ and $\omega_c(d = 3) \simeq 0.24$. The distribution $P(v)$ of the positive random variable v of order $O(1)$ presents a power-law singularity near the origin $P(v) \propto v^a$ with $a(d = 2, 3) > 0$ so that all moments of the surface magnetization are governed by the same power-law decay $(m_L^{surf})^k \propto L^{-x_s}$ with $x_s = \omega_c(1 + a)$ independently of the order k . Our conclusion is thus that this non-linear transfer approach is able to lead to Infinite Disorder scaling, that had been found previously via Monte-Carlo in $d = 2$ [14, 15] and via Strong Disorder RG in $d = 2, 3, 4$ [3–13]. Exactly at criticality, the presence of activated scaling $\ln m_L^{surf} \simeq -L^{\omega_c} v$ means that the linearization of Eq. 12 is typically still valid also at criticality (and not only in the disordered phase), so that the identity $\omega_c = \omega_{DP}(D = d - 1)$ can be understood. The rare cases where this linearization is not valid at criticality is when the positive random variable v happens to be smaller than $1/L^{\omega_c}$. Our conclusion is thus the following :

(i) in the disordered phase and for 'typical' situations exactly at criticality, the linearization of Eq. 12 is valid and coincides with the leading order perturbative scaling analysis [19] : it is thus likely to give exact values for critical exponents, in particular for the exponent ω_c of activated scaling.

(ii) in the ordered phase and for 'rare' situations at criticality, the non-linear terms of the transfer approach plays an important role. Since they come from an uncontrolled approximation, the critical exponents like β_s and x_s that are determined by these non-linear contributions could be different from the exact ones. To judge the accuracy of this approximation, it would be very helpful to compare with other approaches like Quantum Monte-Carlo and Strong

Disorder RG (but up to now, these other approaches have not studied surface properties).

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